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Quantum dynamics of a particle in a plane wave

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Abstract. We discuss quantum dynamics of a particle in a time and space periodic field. We consider a soluble model of a particle in a plane wave, and reduce the quantum problem to a random Riccati equation. We show that solutions of the Riccati equation experience an abrupt change from a periodic to non-periodic behaviour under a variation of classical parameters with respect to the Planck constant. As a consequence the classical results on particle trapping by an electromagnetic wave need a quantum correction.

1. Introduction

It is a basic question in the theory of periodic quantum systems [1] whether the solutions are quasiperiodic or not. The problem is closely related to the stability of these systems. We cannot answer this question through a perturbation expansion because of the small denominators, which are an obstacle to a proof of the quasiperiodicity of the sum. In this paper we discuss a soluble model of stochastic dynamics equivalent to the quantum mechanics. The model allows a classical description of the phenomena which lead to quantum instability. Our model is a particular example of a particle in an electromagnetic field discussed in [2–5]. It is known that the particle is either trapped by the wave (if the wave amplitude is large with respect to its frequency) or moves in a complicated oscillatory way (if the frequency of the wave is sufficiently large).

We consider in our model the following scalar potential

$$V = \frac{m}{4}(\alpha^2-\beta^2)\cos(2x+2\omega t)-\frac{m\alpha\beta}{2}\sin(2x+2\omega t)+m\alpha\omega\sin(x+\omega t)+m\beta\omega\cos(x+\omega t). \tag{1}$$

We can find a particular solution of the classical Hamilton–Jacobi equation

$$\partial_t S_t + \frac{1}{2m}(\nabla S_t)^2 + V = 0. \tag{2}$$

The solution is

$$S_t(x) = m\alpha\cos(x+\omega t) - m\beta\sin(x+\omega t) + \frac{m}{4}(\alpha^2+\beta^2)t. \tag{3}$$

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$S_0(x)$ could be considered as the Hamilton–Jacobi description of the separatrix. The classical equation of motion resulting from solution (3) has the form (it becomes the conventional pendulum's separatrix equation when $\omega = \beta = 0$)

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \alpha \sin(x + \omega t) + \beta \cos(x + \omega t). \tag{4}$$

Equation (4) is explicitly integrable. Let

$$\xi = x + \omega t \qquad \text{and} \qquad u = \tan(\tfrac{1}{2}\xi). \tag{5}$$

Then,

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \tfrac{1}{2}(\omega - \beta)u^2 + \alpha u + \tfrac{1}{2}(\omega + \beta). \tag{6}$$

It is clear from (6) that properties of its solution depend in a crucial way on whether $\alpha^2 + \beta^2 \geqslant \omega^2$ (a particle is trapped) or $\alpha^2 + \beta^2 < \omega^2$ (lack of trapping).

In the first case the particle moves with the wave and for large time the only effect of the nonlinear dynamics is a phase shift (the phase shift tends to a constant with an exponential speed). In the second case the particle oscillates with its own frequency

$$\mu = \sqrt{\omega^2 - \alpha^2 - \beta^2}. \tag{7}$$

2. Stochastic description

In order to investigate the quantum dynamics we apply a probabilistic formulation of the Feynman integral. We write the solution of the Schrödinger equation in the form

$$\psi_t(x) \equiv (U_t\psi)(x) = E\left[\exp\left(-\frac{\mathrm{i}}{\hbar}\int_0^t V(x + \sigma b_s)\,\mathrm{d}s\right)\psi(x + \sigma b_t)\right]. \tag{8}$$

Here, $t \geqslant 0$

$$\sigma = (1 + \mathrm{i})\sqrt{\frac{\hbar}{2m}}$$

and the expectation value is with respect to the Brownian motion defined as the Gaussian process with the covariance

$$E[b(t)b(s)] = \min(t, s). \tag{9}$$

The generator of U_t is

$$\frac{1}{\mathrm{i}\hbar}H = \frac{\mathrm{i}\hbar}{2m}\Delta + \frac{1}{\mathrm{i}\hbar}V. \tag{10}$$

Assume we know a particular (complex) solution S_t of the Hamilton–Jacobi equation (2) with the initial condition S_0. Let us write the initial state ψ in the form

$$\psi = \exp\left(\frac{\mathrm{i}}{\hbar}S_0\right)\phi. \tag{11}$$

Then formula (8) can be expressed as

$$\psi_t(x) = \exp\left(\frac{\mathrm{i}}{\hbar} S_t(x)\right) E\left[\exp\left\{-\frac{1}{2m}\int_0^t \Delta S_\tau(q_\tau(x))\,\mathrm{d}\tau\right\}\phi(q_0(x))\right] \tag{12}$$

where q_τ is the solution of the stochastic equation

$$\mathrm{d}q_\tau = -\frac{1}{m}\nabla S_\tau\,\mathrm{d}\tau + \sigma\,\mathrm{d}b \tag{13}$$

with the boundary condition $q_\tau(x)_{|\tau=t} = x$.

There is another (equivalent) formula which expresses ψ_t by a solution of the stochastic equation (13) with the initial condition

$$q_\tau(x)_{|\tau=0} = x. \tag{14}$$

In such a case $\phi(q_0(x))$ in (6) is replaced by $\phi(q_t(x))$.

We assume that all functions discussed here are boundary values of analytic functions. Then, the formulae (12) and (13) make sense. We discussed a slightly modified version of (12) and (13) in our earlier papers [6, 7]. There, we applied a particular solution of the Schrödinger equation instead of the Hamilton–Jacobi equation. In such a case the exponential factor in the square brackets of (12) is absent.

Note that $\hbar$ enters (12) only through q. The formal classical limit ($\hbar \to 0$) of (12) and (13) is

$$\psi_t(x) = \exp\left(\frac{\mathrm{i}}{\hbar} S_t(x)\right) \exp\left(-\frac{1}{2m}\int_0^t (\Delta S_\tau)(x_\tau(x))\,\mathrm{d}\tau\right)\phi(x_0(x)) \tag{15}$$

where x_τ is the solution of the equation

$$\frac{\mathrm{d}x_\tau}{\mathrm{d}\tau} = -\frac{1}{m}\nabla S_\tau(x_\tau) \tag{16}$$

with the initial condition $x_{\tau_{|\tau=t}} = x$.

We discuss in this paper the model (3) (which by (2) corresponds to the quantum mechanics with the potential (1)). The stochastic equation (13) reads

$$\mathrm{d}q = \alpha\sin(q+\omega t)\mathrm{d}t + \beta\cos(q+\omega t)\,\mathrm{d}t + \sigma\,\mathrm{d}b. \tag{17}$$

Using the exponential representation of the trigonometric functions we can transform (17) into the Riccati equation

$$\frac{\mathrm{d}z}{\mathrm{d}t} = f_0(t) + f_1(t)z + f_2(t)z^2. \tag{18}$$

The Riccati equation can subsequently be linearized by the substitution

$$W(t) = \exp\left(-\int_0^t f_2(s)z(s)\,\mathrm{d}s\right). \tag{19}$$

Then, W fulfils a linear equation

$$\frac{\mathrm{d}^2W}{\mathrm{d}t^2} - \left(f_1(t) + \frac{\mathrm{d}}{\mathrm{d}t}\ln f_2(t)\right)\frac{\mathrm{d}W}{\mathrm{d}t} + f_0(t)f_2(t)W = 0. \tag{20}$$

In the model (17) we may choose

$$z(t) = \exp(-\mathrm{i}\omega t - \mathrm{i}q(t))$$

then

$$f_0 = -\frac{\alpha + \mathrm{i}\beta}{2} \qquad f_2 = \frac{\alpha - \mathrm{i}\beta}{2} \qquad \text{and} \qquad f_1(t) = -\mathrm{i}\left(\omega + \sigma\frac{\mathrm{d}b}{\mathrm{d}t}\right).$$

Hence from (19)

$$W = \exp\left(-\tfrac{1}{2}(\alpha - \mathrm{i}\beta)\int_0^t \exp(-\mathrm{i}\omega s - \mathrm{i}q(s))\,\mathrm{d}s\right). \tag{21}$$

We can also consider

$$z_*(t) = \exp(\mathrm{i}\omega t + \mathrm{i}q(t))$$

then

$$f_0 = -\frac{\alpha - \mathrm{i}\beta}{2} \qquad f_2 = \frac{\alpha + \mathrm{i}\beta}{2} \qquad \text{and} \qquad f_1(t) = \mathrm{i}\left(\omega + \sigma\frac{\mathrm{d}b}{\mathrm{d}t}\right).$$

Hence from (19)

$$W_* = \exp\left(-\tfrac{1}{2}(\alpha + \mathrm{i}\beta)\int_0^t \exp(\mathrm{i}\omega s + \mathrm{i}q(s))\,\mathrm{d}s\right). \tag{22}$$

Both fulfil linear equations

$$\frac{\mathrm{d}^2 W}{\mathrm{d}t^2} + \mathrm{i}\left(\omega + \sigma\frac{\mathrm{d}b}{\mathrm{d}t}\right)\circ\frac{\mathrm{d}W}{\mathrm{d}t} - \tfrac{1}{4}(\alpha^2 + \beta^2)W = 0 \tag{23}$$

whereas W_*

$$\frac{\mathrm{d}^2 W_*}{\mathrm{d}t^2} - \mathrm{i}\left(\omega + \sigma\frac{\mathrm{d}b}{\mathrm{d}t}\right)\circ\frac{\mathrm{d}W_*}{\mathrm{d}t} - \tfrac{1}{4}(\alpha^2 + \beta^2)W_* = 0 \tag{24}$$

where $\circ$ means the Stratonovitch differential [8] (the Stratonovitch form in (24) follows from (17)).

Equations (23) and (24) can be rewritten as a well-defined system of two linear stochastic differential equations. For this purpose let us introduce the notation

$$V^T = \left(W, \frac{\mathrm{d}W}{\mathrm{d}t}\right) \tag{25}$$

then we can write (23) in the form

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \mathcal{A}\circ V \tag{26}$$

where (the multiplication is understood in the Stratonovitch sense)

$$\mathcal{A} = \begin{bmatrix} 0 & 1 \\ \gamma^2 & \nu_b \end{bmatrix} \tag{27}$$

with $\nu_b = -\mathrm{i}(\sigma(\mathrm{d}b/\mathrm{d}t) + \omega)$ and

$$\gamma^2 = \frac{\alpha^2 + \beta^2}{4}. \tag{28}$$

The solution of (27) can be expressed in the form

$$V_t = \mathcal{R}_t V_0 \tag{29}$$

where we write $\mathcal{R}$ as a product

$$\mathcal{R}_t = \mathcal{M}_t \mathcal{D}_t \tag{30}$$

where

$$\begin{aligned} \mathcal{M} &= \begin{bmatrix} 1 & 0 \\ 0 & \rho \end{bmatrix} \\ \rho_t &= \exp \int_0^t \nu_b = \exp(-\mathrm{i}\omega t - \mathrm{i}\sigma b_t) \end{aligned} \tag{31}$$

and $\mathcal{D}_t$ is the solution of the equation (with the initial condition $\mathcal{D}_0 = 1$)

$$\frac{\mathrm{d}\mathcal{D}_t}{\mathrm{d}t} = \mathcal{B}_t \mathcal{D}_t \tag{32}$$

where

$$\mathcal{B} = \mathcal{M}^{-1} \begin{bmatrix} 0 & 1 \\ \gamma^2 & 0 \end{bmatrix} \mathcal{M} \equiv \begin{bmatrix} 0 & \rho \\ \delta & 0 \end{bmatrix} \tag{33}$$

where

$$\delta(t) = \gamma^2 \rho_t^{-1} = \gamma^2 \exp(\mathrm{i}\omega t + \mathrm{i}\sigma b_t). \tag{34}$$

We can solve (32) by iteration (the series is convergent).

3. The effect of quantum noise

At $\sigma = 0$ we would get the classical solution (6). We are interested in the effect of a complex noise on the behaviour of the deterministic solution.

Let us denote $\mathrm{d}W/\mathrm{d}t = W'$, then (26) can be expressed as a system of two Ito equations

$$\begin{aligned} \mathrm{d}W &= W' \,\mathrm{d}t \\ \mathrm{d}W' &= -\mathrm{i}\left(\omega - \frac{\hbar}{2m}\right) W' \,\mathrm{d}t + \gamma^2 W \,\mathrm{d}t - \mathrm{i}\sigma W \,\mathrm{d}b \end{aligned} \tag{35}$$

where the relation $f \circ \mathrm{d}b = f\,\mathrm{d}b + \frac{1}{2}\mathrm{d}f\,\mathrm{d}b$ (see [8]) between Stratonovitch and Ito differentials has been used. Let $w = E[W]$, then w is the solution of a deterministic equation

$$\frac{\mathrm{d}^2 w}{\mathrm{d}t^2} + \mathrm{i}\left(\omega - \frac{\hbar}{2m}\right)\frac{\mathrm{d}w}{\mathrm{d}t} - \gamma^2 w = 0. \tag{36}$$

In order to derive (36) we write (35) in the Ito integral form, then we take the expectation value and use the property of the Ito integral

$$E\left[\int f\,\mathrm{d}b_s\right] = 0.$$

Rewriting the resulting integral equation for the expectation value $E[W]$ again in a differential form we obtain (36). We can see from (36) that the only effect of noise on the expectation value is a change of the frequency

$$\omega \to \omega - \frac{\hbar}{2m}.$$

Hence, from (7) we can see (γ is defined in (28)) that if

$$|\omega^2 - 4\gamma^2| > \left|\frac{\omega\hbar}{m} - \frac{\hbar^2}{4m^2}\right| \tag{37}$$

then the quantum noise has no effect on stability, changing only the frequency (or the Lyapunov exponent). However, if condition (37) is not fulfilled, then the noise can change the exponential increase of $w(t)$ into an oscillatory behaviour or vice versa. So, if

$$|\omega| > 2\gamma > \left||\omega| - \frac{\hbar}{2m}\right| \tag{38}$$

then the classical variable $W(t)$ ((23) at $\sigma = 0$) is periodic whereas the quantum expectation value $E[W]$ grows exponentially. If

$$\left||\omega| + \frac{\hbar}{2m}\right| > 2\gamma > |\omega| \tag{39}$$

then the classical variable grows exponentially whereas the quantum expectation value is periodic. We can conclude that quantum effects are important for particle trapping near the critical value of the wave frequency $|\omega| = 2\gamma$.

The explicit form of the solution of (36) is

$$w(t) = A(t)w(0) + C(t)\frac{\mathrm{d}w}{\mathrm{d}t}(0) \tag{40}$$

where

$$A = \frac{1}{\nu}[\nu_+ \exp(\mathrm{i}\nu_- t) - \nu_- \exp(\mathrm{i}\nu_+ t)] \tag{41}$$

and

$$C = -\frac{1}{\nu}[\exp(\mathrm{i}\nu_+ t) - \exp(\mathrm{i}\nu_- t)] \tag{42}$$

with

$$\nu_\pm = \tfrac{1}{2}(\nu_0 \pm \nu) \tag{43}$$

where

$$\nu_0 = \omega - \frac{\hbar}{2m} \qquad \text{and} \qquad \nu = \sqrt{\nu_0^2 - 4\gamma^2}. \tag{44}$$

Formulae (40)–(44) are interesting from the point of view of an expansion in γ in (32) and (33) and in the Feynman formula (8) (the Dyson expansion). We can see by the comparison of the expansion in γ of $w(t)$ (equation (40)) with (32) and (33) that ν_0 will be the small denominator in the expansion. The appearance of the small denominator indicates a qualitative change of the dynamics when $\nu_0 \to 0$. It is clear that this structural change is the trapped–untrapped transition discussed above.

So far, we have computed the expectation value of W and not of q or $\cos q$, which are the physically meaningful variables. Although q can be expressed by W from (21), the relation between these variables is nonlinear, therefore we need all correlation functions of W in order to compute the correlations of q. For this purpose we need to study the effect of noise on higher-order correlation functions. The two-point function is

$$E[W(t)W(s)] = E[W(t,s)]E[W(s)] + E[W(s)^2] \tag{45}$$

where $W(t,s)$ means that the initial condition in (23) is at s instead of 0. We used the additivity of the integral in the derivation of (45) and the independence of increments of the Brownian motion. $E[W(t,s)]$ can be computed from (36). It remains to compute the second term in (45). For this purpose let us compute

$$\begin{aligned} \mathrm{d}(W^2) &= 2W\,\mathrm{d}W = 2WW'\,\mathrm{d}t \\ \mathrm{d}(WW') &= \mathrm{d}WW' + W\,\mathrm{d}W' \\ \mathrm{d}(W'W') &= 2W'\,\mathrm{d}W' + \mathrm{d}W'\,\mathrm{d}W'. \end{aligned} \tag{46}$$

Then, we use the stochastic equations (35) in order to express the differentials of W' in terms of W and W'. We integrate (46) and subsequently take the expectation value. As a result we obtain a system of three equations for the expectation values $w_{22} = E[W'W']$, $w_{11} = E[W^2]$ and $w_{12} = E[WW']$. The equations can be expressed in differential form as an equation for the vector $\mathcal{W}^T = (w_{22}, w_{12}, w_{11})$

$$\frac{\mathrm{d}\mathcal{W}}{\mathrm{d}t} = \Lambda\mathcal{W} \tag{47}$$

where

$$\Lambda = \begin{bmatrix} M & 2\gamma^2 & 0 \\ 1 & N & \gamma^2 \\ 0 & 2 & 0 \end{bmatrix} \tag{48}$$

with

$$M = -2\mathrm{i}\left(\omega + \frac{\hbar}{4m}\right) \qquad N = -\mathrm{i}\left(\omega - \frac{\hbar}{2m}\right).$$

Equation (47) is equivalent to a third-order equation with constant coefficients. If the inequality (37) is fulfilled, then the quantum corrections do not change the classical behaviour ($w_{11} = w(\hbar = 0)^2$); if not then the quantum corrections can change the periodic behaviour into a non-periodic one. We could continue the method to compute correlation functions of arbitrary order. The problem is reduced to linear ordinary differential equations of an increasing order.

4. Change of variables in the Feynman integral

We wish to rewrite the Feynman formula (12) in terms of the W variables. We can easily check that in the model (3)

$$W(t)W_*(t) = \exp\left(\frac{1}{m}\int_0^t \Delta S_\tau(q(\tau))\,\mathrm{d}\tau\right). \tag{49}$$

Hence, the Feynman formula can be expressed in the form

$$(U_t\psi)(x) = \exp\left(\frac{\mathrm{i}}{\hbar}S_t(x)\right) E\left[\left(\sqrt{W(t)W_*(t)}\right)^{-1}\phi(q_t(W))\right] \tag{50}$$

assuming that we are able to define the square root unambiguously (the right-hand side is well defined for small t, when W is close to 1; the square root can be defined uniquely from 1 by the requirement of the semigroup composition law for U_t).

We are interested now in the variable WW_* in (50). Using (23) and (24) we can derive an equation for WW_* in the same way as we obtained one for W^2. Denote $\zeta^T = (WW_*, WW_*', W'W_*, W'W_*')$ then ζ fulfils the following Stratonovitch stochastic differential equation

$$\frac{\mathrm{d}\zeta}{\mathrm{d}t} = \Theta \circ \zeta \tag{51}$$

where

$$\Theta = \begin{bmatrix} 0 & 1 & 1 & 0 \\ \gamma^2 & -\nu_b & 0 & 1 \\ \gamma^2 & 0 & \nu_b & 1 \\ 0 & \gamma^2 & \gamma^2 & 0 \end{bmatrix} \tag{52}$$

with

$$\nu_b = -\mathrm{i}\omega - \mathrm{i}\sigma\frac{\mathrm{d}b}{\mathrm{d}t}.$$

For a discussion of solutions of (52) it is useful to decompose the matrix Θ

$$\Theta = \Theta_c + B \tag{53}$$

where

$$\Theta_c = \begin{bmatrix} 0 & 1 & 1 & 0 \\ \gamma^2 & \mathrm{i}\omega & 0 & 1 \\ \gamma^2 & 0 & -\mathrm{i}\omega & 1 \\ 0 & \gamma^2 & \gamma^2 & 0 \end{bmatrix} \tag{54}$$

and

$$\mathcal{B} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & B & 0 & 0 \\ 0 & 0 & -B & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{55}$$

where

$$B = \mathrm{i}\sigma \frac{\mathrm{d}b}{\mathrm{d}t}.$$

Now, the solution of (51) can be expressed in the form

$$\zeta_t = \mathcal{F}_t \zeta_0 \tag{56}$$

where

$$\mathcal{F}_t = \exp(t\Theta_c)\mathcal{G}_t \tag{57}$$

where $\mathcal{G}_t$ is the solution of the equation (with the initial condition $\mathcal{G}_0 = 1$)

$$\frac{\mathrm{d}\mathcal{G}}{\mathrm{d}t} = \exp(-t\Theta_c)\mathcal{B}\exp(t\Theta_c)\mathcal{G}. \tag{58}$$

Equation (58) can be used for a successive expansion in $\hbar$. The exponentiation of Θ_c is equivalent to the problem of solving a system of four linear differential equations with constant coefficients. This can be done in an elementary way. It is sufficient to use the solution (40) of (23) and (24) at $\hbar = 0$. Inserting the initial conditions resulting from (14) we obtain

$$\begin{aligned} W^c = {} & -\frac{\alpha - \mathrm{i}\beta}{\mu} \exp\left(-\mathrm{i}x - \frac{\mathrm{i}\omega t}{2}\right) \sin(\tfrac{1}{2}\mu t) - \mu^{-1} \\ & \times \exp\left(-\frac{\mathrm{i}\omega t}{2}\right)\left(\mu_- \exp\left(-\frac{\mathrm{i}\mu t}{2}\right) - \mu_+ \exp\left(\frac{\mathrm{i}\mu t}{2}\right)\right) \end{aligned} \tag{59}$$

where

$$\mu = \sqrt{\omega^2 - 4\gamma^2} \qquad \text{and} \qquad \mu_\pm = \frac{\omega}{2} \pm \frac{\mu}{2}.$$

Then, $W^c_* = \overline{W^c}$ and

$$\begin{aligned} W(t)W_*(t)_{|\hbar=0} &= |W^c|^2 \\ &= \mu^{-2}\left|(\alpha - \mathrm{i}\beta)\sin(\tfrac{1}{2}\mu t) + \exp(\mathrm{i}x)\left(\mu_- \exp\left(-\frac{\mathrm{i}}{2}\mu t\right) - \mu_+ \exp\left(\frac{\mathrm{i}}{2}\mu t\right)\right)\right|^2. \end{aligned} \tag{60}$$

Equations (50), (59) and (60) give the semiclassical formula (15) for the motion of a quantum particle in a plane wave.

Note that from (19)

$$\exp(-\mathrm{i}q(s) - \mathrm{i}\omega s) = -2(\alpha - \mathrm{i}\beta)^{-1} W'(s) W(s)^{-1}. \tag{61}$$

From (59) and (61) we can get the classical solution $q_c(s)$ (6) describing the trapped (if $2\gamma \geqslant \omega$) or untrapped (if $2\gamma < \omega$) motion of a particle. If the initial wavefunction ψ is localized at the origin, for example

$$\psi(x) = \exp\left(\frac{\mathrm{i}S(x)}{\hbar}\right) \exp\left(-\frac{x^2}{2\delta}\right)$$

then the trapping is expressed by the localization of ψ_t at $x = -\omega t$. Moreover, the exponential behaviour of W_c expresses the exponential decay of $x + \omega t$ to a constant. The lack of trapping means that $\psi_t(x)$ remains localized at the origin and the probability density $|\psi_t(x)|^2$ varies periodically with the frequency μ. This picture must be corrected by the caustic singularities corresponding to a zero of W_c. $W_c = 0$ if

$$\exp(-\mathrm{i}x) = (\alpha + \mathrm{i}\beta)^{-1} \left(\mathrm{i}\omega + \mu \cot\left(\frac{\mu t}{2}\right)\right).$$

This equation has a solution only if the particle is trapped ($\mu^2 < 0$).

We can find the matrix $\exp(t\Theta_c)$ (equation (54)) by differentiation of $W_c(t)$ and $\overline{W_c}(t)$. Then, we obtain a convergent expansion in $\hbar$ of $\mathcal{F}$ (equations (56)–(58)); in particular, an expansion of WW_*.

In order to compute the expectation value of $(\sqrt{WW_*(t)})^{-1}$ we can again use an expansion in $\hbar$. For this purpose we compute $E[(WW_*(t))^n]$. The method is the same as in the computation of $E[W(t)^n]$. For $n = 1$ denote

$$X_1 = WW'_* - W_*W' \qquad X_2 = WW'_* - W_*W' \qquad X_3 = WW_* \qquad X_4 = W'W'_*.$$

We obtain for the vector X the equation

$$\frac{\mathrm{d}X}{\mathrm{d}t} = \Omega \circ X \tag{62}$$

where

$$\Omega = \begin{bmatrix} 0 & -\nu_b & 0 & 0 \\ -\nu_b & 0 & 2\gamma^2 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & \gamma^2 & 0 & 0 \end{bmatrix}. \tag{63}$$

From (63) we can compute $E[WW_*]$. The result is that the classical frequencies $\mu_\pm$ in (60) are shifted by $\omega \to \omega \pm \hbar/2m$. We can use the solution of (62) to compute $(WW_*)^2$ and continue the procedure to higher orders.

We return finally to the main result of this paper discussed in section 3. So, when $\omega - 2\gamma$ is small a change of the frequency of order $\hbar/m$ can have a drastic effect on the large time behaviour of the variable W. We would like to estimate the effect of the change of behaviour of W on the behaviour of the wavefunction ψ_t (50). It is not easy to see this effect from an expansion in $\hbar$ around the classical solution (59). We consider another expansion. We may restrict ourselves to periodic functions $\phi(x)$ in (50). Expanding ϕ in a Fourier series we obtain from (61)

$$\phi(q_t(x)) = \sum a_n \exp(-inq_t(x)) = \sum_{n>0} a_n \exp(in\omega t)(-2)^n(\alpha - i\beta)^{-n}\left(\frac{W'}{W}\right)^n$$
$$+ \sum_{n\leqslant 0} a_n \exp(-in\omega t)(-2)^n(\alpha + i\beta)^{-n}\left(\frac{W'_*}{W_*}\right)^n. \tag{64}$$

In section 3 we have shown that $w \equiv E[W]$ and $w' \equiv E[W']$ (as well as w_* and w'_*) undergo a transition from periodic to non-periodic behaviour if $\omega - 2\gamma \simeq \hbar/2m$. Moreover,

$$w = E[W(\omega)] = W_c\left(\omega - \frac{\hbar}{2m}\right) \tag{65}$$

(and a similar formula for w', w_* and w'_*). Hence, we obtain immediately the effect of the quantum noise on trapping if we make, in (50) and (64), an approximation

$$E\left[\left(\frac{W'}{W}\right)^n\left(\sqrt{WW_*}\right)^{-1}\right] \simeq \frac{E[W']^n}{E[W]^n}\left(\sqrt{E[W]E[W_*]}\right)^{-1}. \tag{66}$$

Then, from (65), the effect of the change of the behaviour of $E[W]$ on the behaviour of $\psi_t(x)$ is the same as in the semiclassical approximation discussed after (59) and (60).

In order to investigate an error of the approximation (66) let us write

$$W = w + (W - w) \equiv w + Q = w\left(1 + \frac{Q}{w}\right). \tag{67}$$

Then

$$\frac{W'}{W} = \frac{w'}{w}\left(1 + \frac{Q'}{w'}\right)\left(1 + \frac{Q}{w}\right)^{-1}. \tag{68}$$

Q fulfils the stochastic differential equation

$$\frac{d^2Q}{dt^2} + i\left(\omega + \sigma\frac{db}{dt}\right)\circ\frac{dQ}{dt} - \gamma^2 Q = -\frac{i\hbar w'}{2m} - i\sigma w' \circ \frac{db}{dt} \tag{69}$$

with the initial condition $Q(0) = Q'(0) = 0$.

It can be seen from (69) that Q is of order $\hbar$. Moreover, we can rewrite (69) as a system of first-order equations and express their solution in terms of the random matrix $\mathcal{R}_t$ defined in (30)–(34). Then W'/W in (64) expressed by Q in (68) is of the form

$$\frac{w'}{w}(1 + \hbar F(\mathcal{R}))(1 + \hbar G(\mathcal{R}))^{-1} \tag{70}$$

where F and G are certain functionals which we do not determine here. We expect that (66) is a good approximation in the sense that the corrections expressed symbolically in (70) are of order $\hbar$ uniformly in time if $\mathcal{R}$ behaves in an oscillatory way (then the particle remains untrapped). The transition to the exponential behaviour of $\mathcal{R}$ occurs (as follows from the discussion in section 3) at $\omega - 2\gamma \simeq \hbar/2m$. We can still apply formula (70) because the nominator and the denominator increase at the same rate. Equation (70) when inserted into (64) leads to the same conclusion in the case of an exponential growth of W as the approximation (66), i.e. that the particle is trapped if $|\omega - \hbar/2m| \leqslant 2\gamma$. In principle, we could check our arguments inserting (68) into (64) and computing the expectation values in (50) of powers of Q and Q' resulting from an expansion of the denominator. At each order the computation of the expectation values is reduced to a system of ordinary differential equations with constant coefficients (as in equation (47)). The difficulty of solving such a system of equations increases rapidly with the order. Nevertheless, we should be able to prove at each order that the transition from the periodic to aperiodic behaviour of expectation values of powers of Q and Q' occurs at $\omega = 2\gamma + \hbar/2m + o(\hbar)$.

Summarizing, we have shown that the random variable W linearizing the quantum mechanical model of a particle in a plane wave can undergo an abrupt change of behaviour. This occurs when the wave amplitude is close to the classical critical value corresponding to the particle transition between trapped and untrapped states. Then, quantum effects of order $\hbar/m$ can cause the transition. The behaviour of variables built from W is well approximated by a convergent expansion in $\hbar$. We gave some arguments showing that the change of the behaviour of the variable W linearizing the model implies a qualitative change of the behaviour of the wavefunction $\psi_t(x)$ corresponding to a quantum description of the trapped–untrapped transition. The model sheds some light also on the meaning of small denominators in quantum mechanics.

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